An Alternative Proof of the Banach-Zarecki Theorem

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Abstract: In this note, we present an alternative proof of the classical theorem by Banach and Zarecki, utilizing the Vitali covering lemma.

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A real-valued function defined over a set of real numbers adheres to Lusin's condition (N) if it maps sets of Lebesgue measure zero to other sets with Lebesgue measure zero. The esteemed Banach-Zarecki theorem (See [4, Chapter IX, Section 3, Theorem 4]) is stated as follows:

Theorem 1 (Banach and Zarecki). *Let F be a real-valued function defined on a bounded closed interval [a,b]. Then F is an absolutely continuous function if and only if F is continuous, possesses bounded variation, and meets Lusin's condition (N).*

This theorem, foundational in real analysis, has broad applications in geometry, functional analysis, and even in various scientific and engineering fields. While its necessity is self-evident, ensuring its sufficiency has always been challenging. We refer the reader to [1, 2, 3] for some contemporary discussions. Subsequently, we offer a proof grounded in the Vitali covering lemma (see [5, Section 6.2]).

Given any positive integer *K*, a real number *X*, and a positive value *r*, we define

$$k * [x - r, x + r] = [x - kr, x + kr].$$

The following proposition can be readily derived using the Vitali covering lemma.

Proposition 2. *Let F be a real Lebesgue measurable function on a bounded closed interval [a,b]. Assuming F is differentiable at any point in the Lebesgue measurable subset E , the following relation holds:*

$$m^*\big(F(E)\big) \leq \int_E |F'|\, dm. \tag{1}$$

Proof. Without loss of generality, suppose that $E \subseteq (a, b)$. For any given positive value M, define the set E_M as $E_M = \{x \in E: |F'(x)| < M\}$. We claim that

$$\mathcal{V} = \{[c,d] \subseteq [a,b]: |\frac{m^*\big(F([c,d])\big)}{d-c}| \leq M, |\frac{m^*\big(F(5 * [c,d])\big)}{5(d-c)}| \leq M\}$$

serves as a Vitali covering for E_M. Indeed, for any $x \in E_M$ and $\epsilon > 0$, there's $\delta > 0$ such that $|F(x) - F(x')| < M|x - x'|$ whenever $|x - x'| \leq \delta$. Assuming $\delta < \min(\epsilon, b - x, x - a)$, we can select the interval $[c,d] = \left[x - \frac{\delta}{5}, x + \frac{\delta}{5}\right]$. For all $x' \in [c,d]$, it holds true that

$$|F(x) - F(x')| \leq M|x - x'| \leq \frac{M(d-c)}{2}.$$

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It follows that $F([c,d]) \subseteq \left[F(x) - \frac{M(d-c)}{2}, F(x) + \frac{M(d-c)}{2}\right]$ *and that* $\left|\frac{m^*(F([c,d]))}{d-c}\right| \leq M$. *An analogous argument shows that* $\left|\frac{m^*(F(5*[c,d]))}{5(d-c)}\right| \leq M$, *which validates our claim.*

From the proof of the Vitali covering lemma (see e.g. [5, P109-110]), for any chosen $\delta > 0$, *there are countable, non-overlapping intervals* $\{I_k\}$ *within* $\mathcal{V}$ *such that* $\sum_k m(I_k) \leq m(E_M) + \delta$, *and the inclusion relation*

$$E_M \subseteq \bigcup_{k=1}^{N} I_k \cup \bigcup_{k=N+1}^{\infty} 5 * I_k$$

is valid for every positive integer N. Hence, for any such N,

$$\begin{aligned} m^*(F(E_M)) &\leq \sum_{k=1}^{N} m^*(F(I_k)) + \sum_{k=N+1}^{\infty} m^*(F(5 * I_k)) \\ &\leq \sum_{k=1}^{N} M\, m(I_k) + \sum_{k=N+1}^{\infty} 5\, Mm(I_k) \leq M(m(E_M) + \delta) + 5M \sum_{k=N+1}^{\infty} m(I_k). \end{aligned}$$

As N approaches infinity and δ *approaches zero, it's deduced that for* $M > 0$,

$$m^*(F(E_M) \leq Mm(E_M).$$

We now proceed to validate Equation (1). For a given $\epsilon > 0$ *and any nonnegative integer n, set* $E_n = \{x \in E: n\epsilon \leq |F'(x)| < (n+1)\epsilon\}$. *It follows that*

$$\begin{aligned} m^*(F(E)) \leq \sum_{n=0}^{\infty} m^*(F(E_n)) &\leq \sum_{n=0}^{\infty} (n+1)\, \epsilon m(E_n) \\ &\leq \sum_{n=0}^{\infty} \left(\int_{E_n} |F'|\, dm + \epsilon m(E_n) \right) \\ &= \int_E |F'|\, dm + \epsilon m(E). \end{aligned}$$

As ϵ *approaches zero, it's observed that* $m^*(F(E)) \leq \int_E |F'|\, dm$.

Now we turn to the proof of the Banach-Zarecki theorem.

the proof of Theorem 1. We will demonstrate sufficiency. Initially, let's assert that for any Lebesgue measurable subset E, the inequality

$$m^*(F(E)) \leq \int_E |F'|\, dm$$

holds. To support this, consider $\widetilde{E}$ *as the subset of E where the derivative* $F'(x)$ *exists. It follows that* $m(E \setminus \widetilde{E}) = 0$. *Consequently, by invoking Lusin's condition (N),* $m(F(E \setminus E')) = 0$. *From Proposition 2, this implies that*

$$m^*(F(E)) \leq m^*(F(\widetilde{E})) + m^*(F(E \setminus \widetilde{E})) \leq \int_{\widetilde{E}} |F'|\, dm = \int_E |F'|\, dm,$$

which validates our assertion.

Coupled with F being continuous, it becomes evident that for every closed subinterval [c,d] of [a,b],

$$|F(c) - F(d)| \leq m(F([c,d])) \leq \int_{[c,d]} |F'|\, dm.$$

Acknowledging F' *belongs to* $L^1[a,b]$, *for a given* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\int_e |F'|\, dm < \epsilon$ *for any Lebesgue measurable subset e where* $m(e) < \delta$. *Thus, for any collection of disjoint closed intervals* $\{[c_i, d_i]\}_{i=1}^{n}$ *with the condition* $\sum_{i=1}^{n} (d_i - c_i) < \delta$, *we have that*

$$\sum_{i=1}^{n} |F(c_i) - F(d_i)| \leq \sum_{i=1}^{n} \int_{[c_i,d_i]} |F'|\, dm \leq \int_{\bigcup_{1 \leq i \leq n} [c_i,d_i]} |F'|\, dm < \epsilon.$$

From this, we infer that F is absolutely continuous.

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